Exact Nonparametric Decentralized Online Optimization

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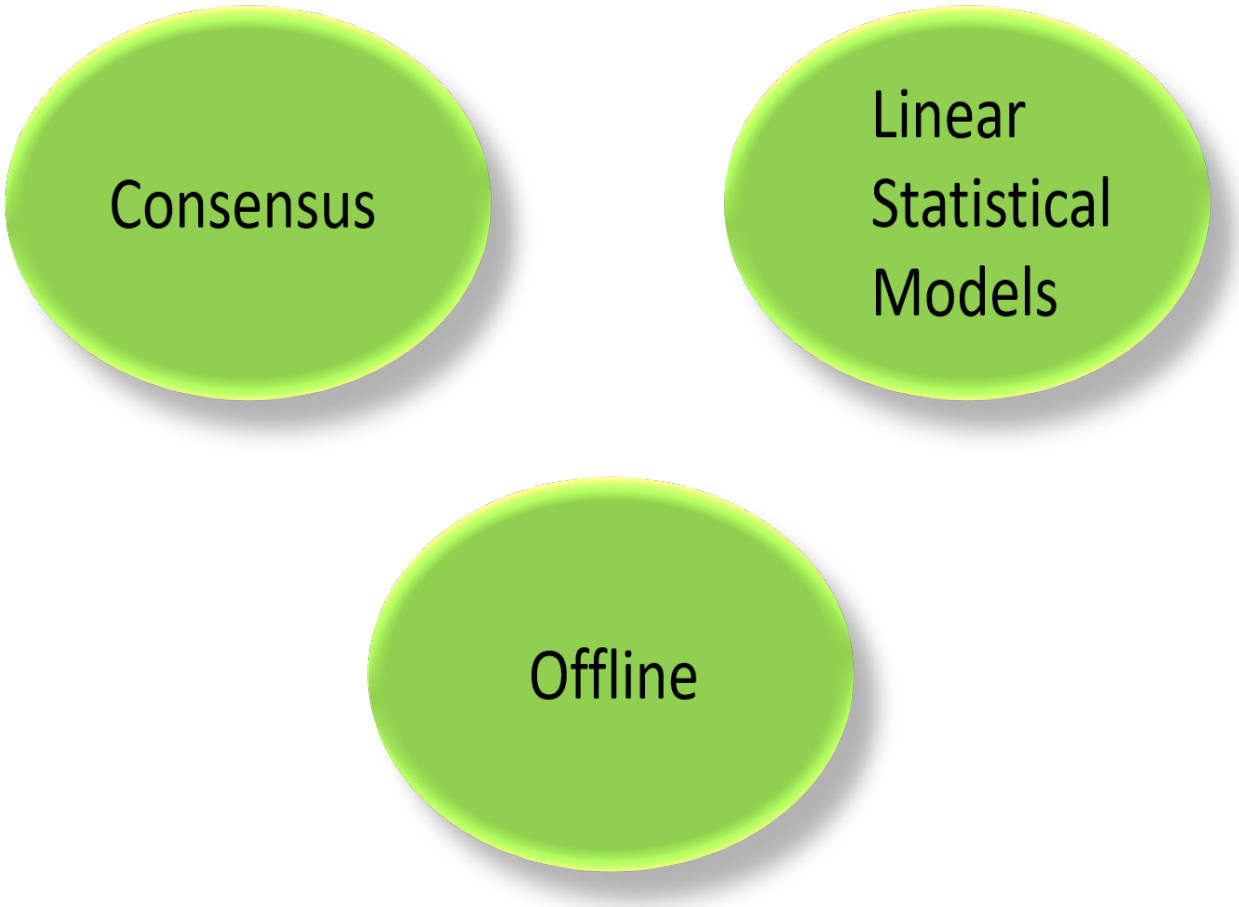
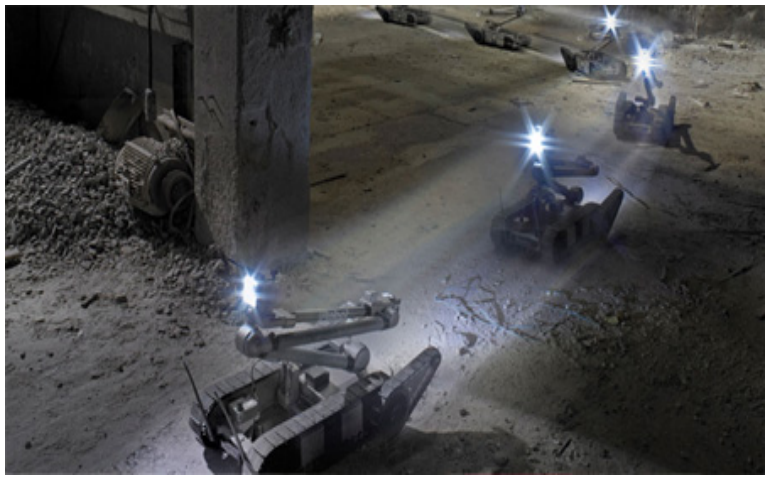
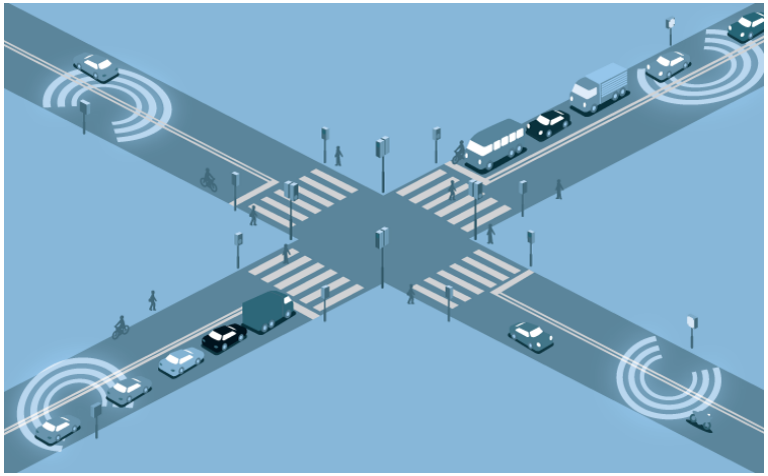
The State of Distributed Learning

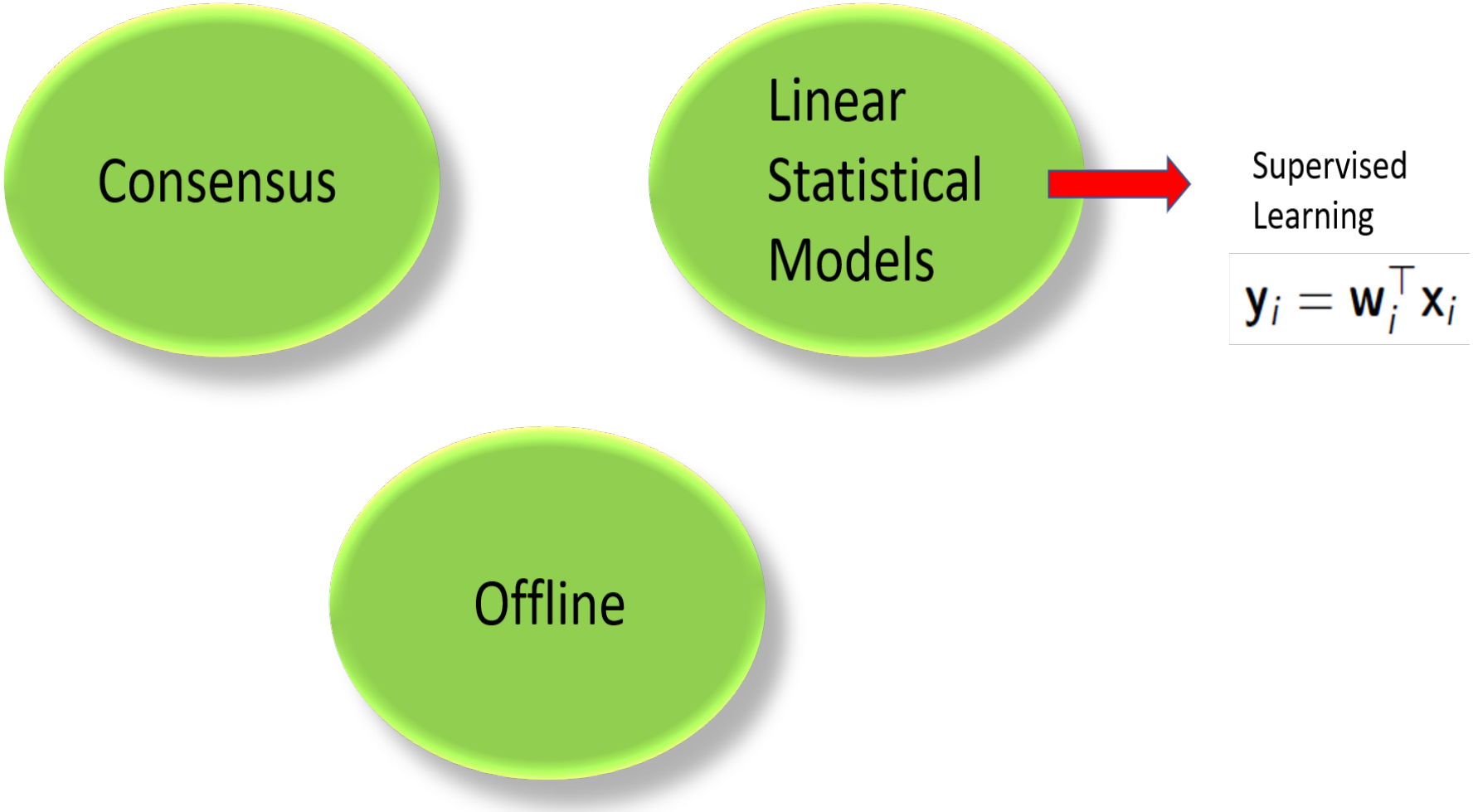


Non-linear Heterogeneous Networked Learning





Existing methods don't apply



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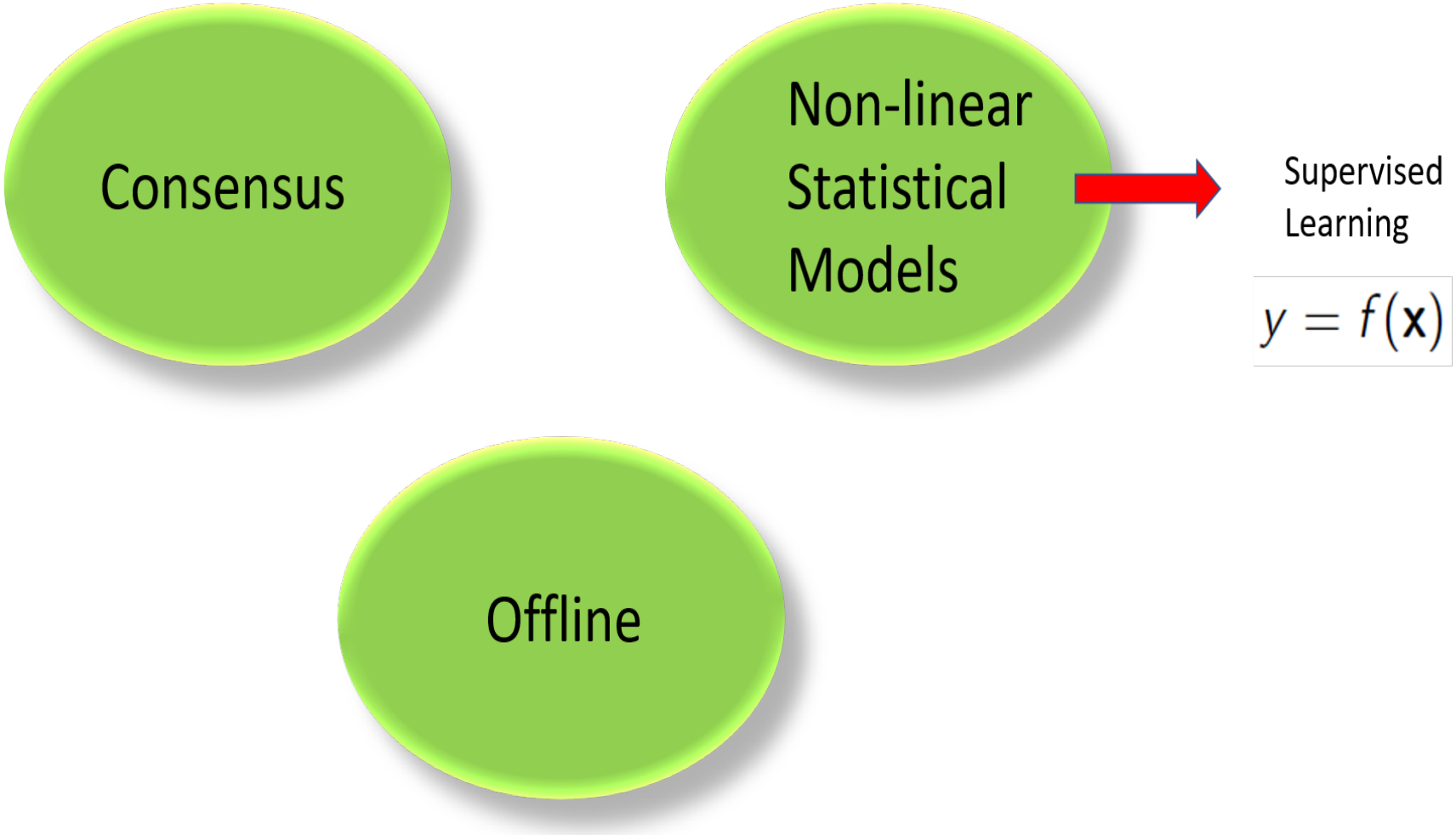

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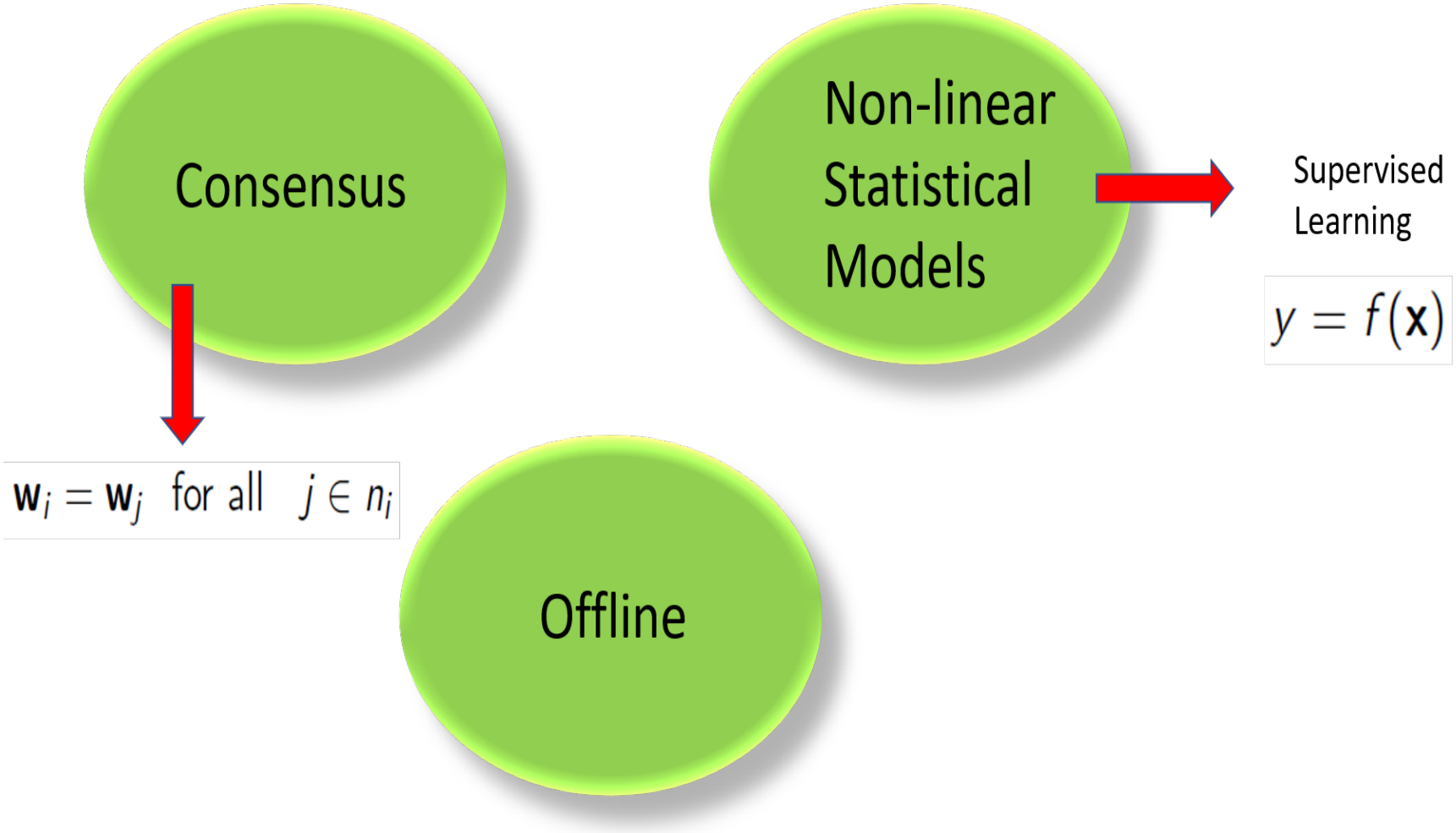

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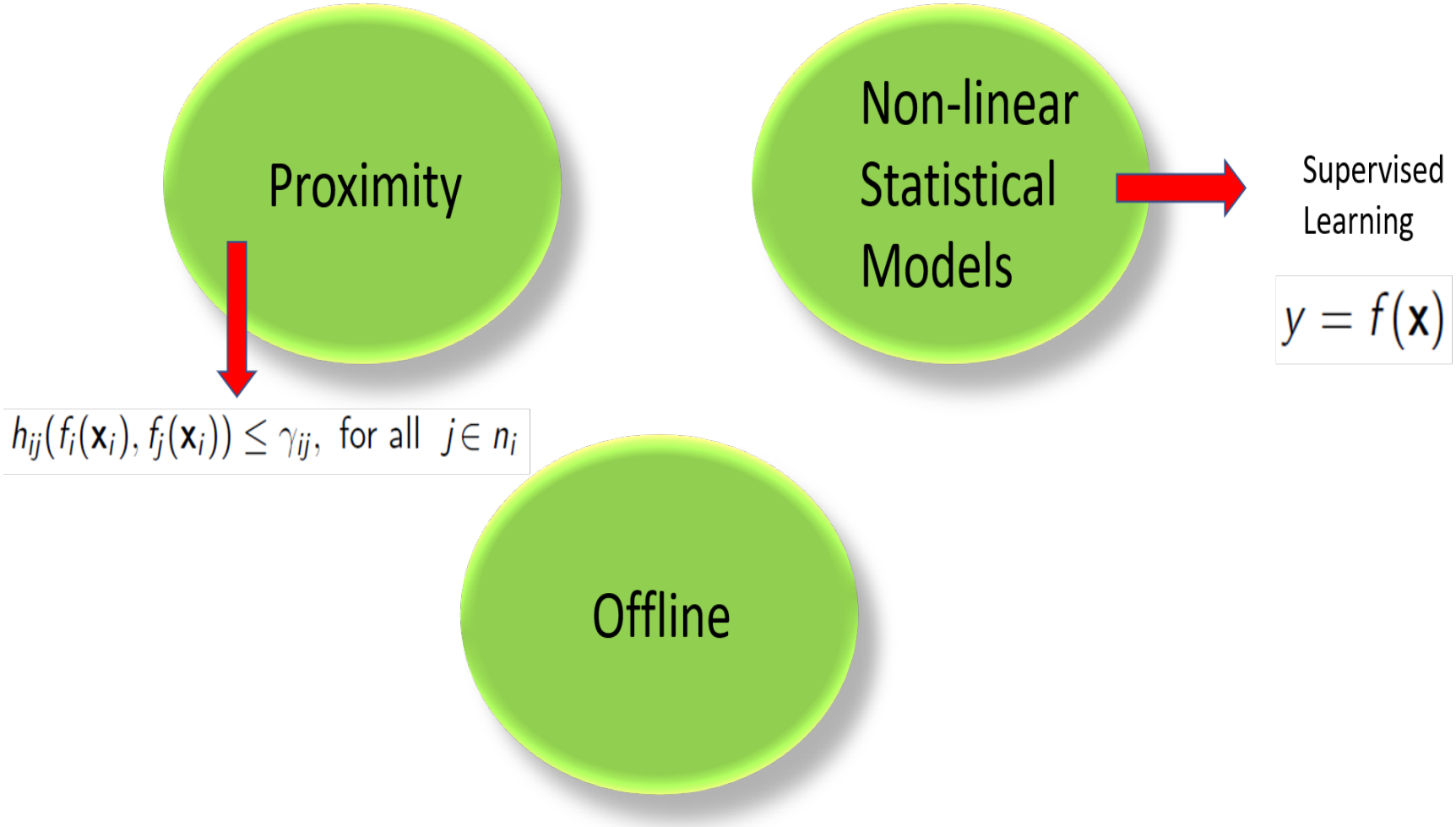

Existing methods don't apply

Proximity

$$h_{ij}(f_i(\mathbf{x}_i), f_j(\mathbf{x}_i)) \leq \gamma_{ij}, \text{ for all } j \in n_i$$

Non-linear Statistical Models → Supervised Learning

$$y = f(\mathbf{x})$$

Offline

$$\mathbf{w}^* := \underset{\mathbf{w} \in \mathbb{R}^{p|\mathcal{V}|}}{\operatorname{argmin}} \frac{1}{N} \sum_{i=1}^{|\mathcal{V}|} \sum_{n=1}^{N} \ell_i(\mathbf{w}_i^\top \mathbf{x}_{i,n}, \mathbf{y}_{i,n})$$

Online Non-linear Heterogeneous Learning

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$$h_{ij}(f_i(\mathbf{x}_i), f_j(\mathbf{x}_i)) \leq \gamma_{ij}, \text{ for all } j \in n_i$$

Non-linear Statistical Models

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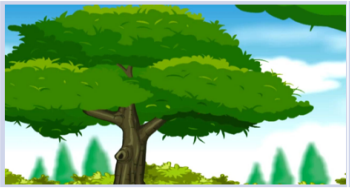
$$y = f(\mathbf{x})$$

Online

$$f^\star = \underset{\{f_i\} \in \mathcal{H}}{\operatorname{argmin}} \sum_{i \in \mathcal{V}} \mathbb{E}_{\mathbf{x}_i, y_i} \left[\ell_i(f_i(\mathbf{x}_i), y_i) \right]$$

Contribution

Distributed Online Learning	Linear	Nonlinear
Homogeneous		Koppel ,
Heterogeneous	Sayed, Chen, Nassif, Lee, Sadler	**Our Work**

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Outline

- **Approach**:
  - $\Rightarrow$ Hypothesized non-linear function in kernel Hilbert space
  - $\Rightarrow$ Form stochastic lagrangian
  - $\Rightarrow$ Apply stochastic primal dual method
  - $\Rightarrow$ Take subspace projection (to handle memory growth)
- Sublinear convergence
  - $\Rightarrow$ $\mathcal{O}(\sqrt{T})$ for primal optimality
  - $\Rightarrow$ $\mathcal{O}(T^{3/4})$ for constraint violation
- Generalizes existing rate results for primal-dual method
  - $\Rightarrow$ to case of non-linear statistical models
- **Application**: Decentralized correlated random field estimation
  - $\Rightarrow$ online non-linear observation model.

The Problem

- Symmetric, connected and directed network of agents $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- Learning nonlinear statistical models is equivalent to finding

  $\Rightarrow f : \mathcal{X} \to \mathcal{Y}$, such that $y = f(\mathbf{x})$
- Loss $\ell : \mathcal{H} \times \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ penalize deviations between $f(\mathbf{x})$, $\mathbf{y}$
- Encoded by a convex proximity function $h_{ij}(f_i(\mathbf{x}_i), f_j(\mathbf{x}_i))$

  $\Rightarrow$ incentivizes nearby agents to make similar decisions
- Yields the constrained functional stochastic program:

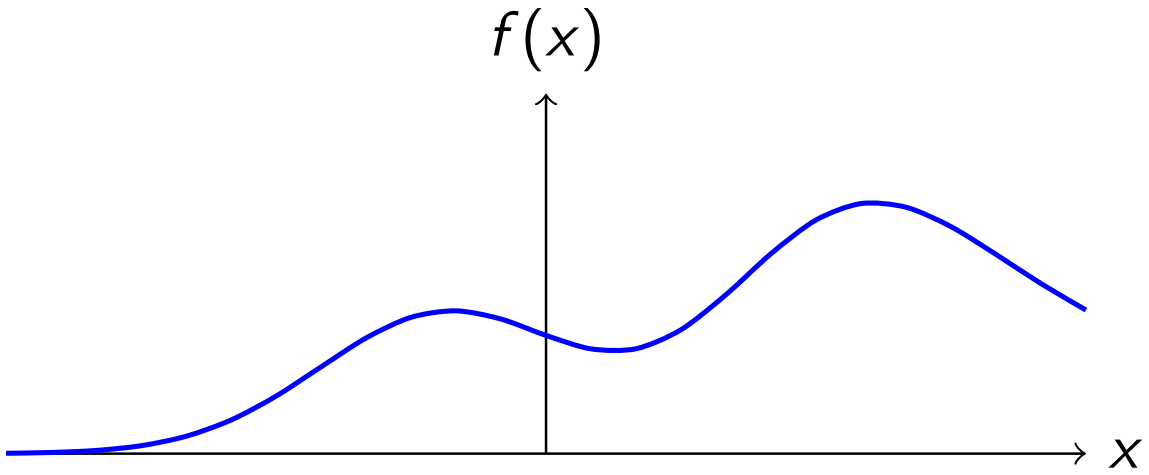
$$f^\star = \underset{\{f_i\} \in \mathcal{H}}{\operatorname{argmin}} \sum_{i \in \mathcal{V}} \left(\mathbb{E}_{\mathbf{x}_i, y_i} \left[\ell_i(f_i(\mathbf{x}_i), y_i) \right] + \frac{\lambda}{2} \|f_i\|^2_{\mathcal{H}} \right)$$

$$\text{s.t.} \quad \mathbb{E}_{\mathbf{x}_i} \left[h_{ij}(f_i(\mathbf{x}_i), f_j(\mathbf{x}_i)) \right] \leq \gamma_{ij}, \ \text{for all} \ \ j \in n_i. \tag{1}$$

Large-Scale Function Estimation

- Equip $\mathcal{H}$ with a unique *kernel function*, $\kappa : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, such that:

$$(i)\ \langle f, \kappa(\mathbf{x}, \cdot)\rangle_{\mathcal{H}} = f(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathcal{X}\ ,$$
$$(ii)\ \mathcal{H} = \text{span}\{\kappa(\mathbf{x}, \cdot)\} \quad \text{for all } \mathbf{x} \in \mathcal{X}\ .$$



- Property (i) $\Rightarrow$ Will allow us to compute derivatives
- Kernel examples:
  - $\Rightarrow$ Gaussian/RBF $\kappa(\mathbf{x}, \mathbf{x}') = \exp\left\{-\frac{\|\mathbf{x}-\mathbf{x}'\|_2^2}{2c^2}\right\}$
  - $\Rightarrow$ polynomial $\kappa(\mathbf{x}, \mathbf{x}') = \left(\mathbf{x}^T\mathbf{x}' + b\right)^c$

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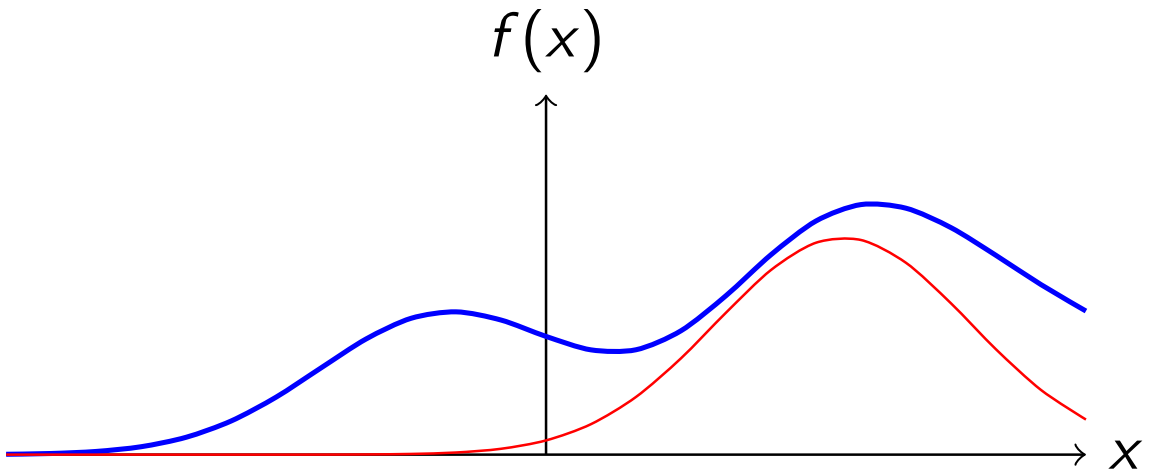

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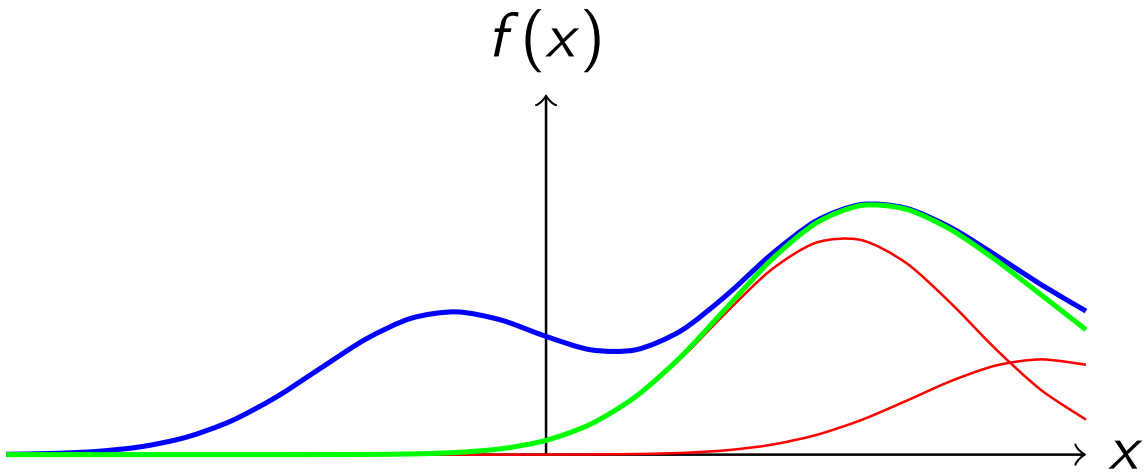

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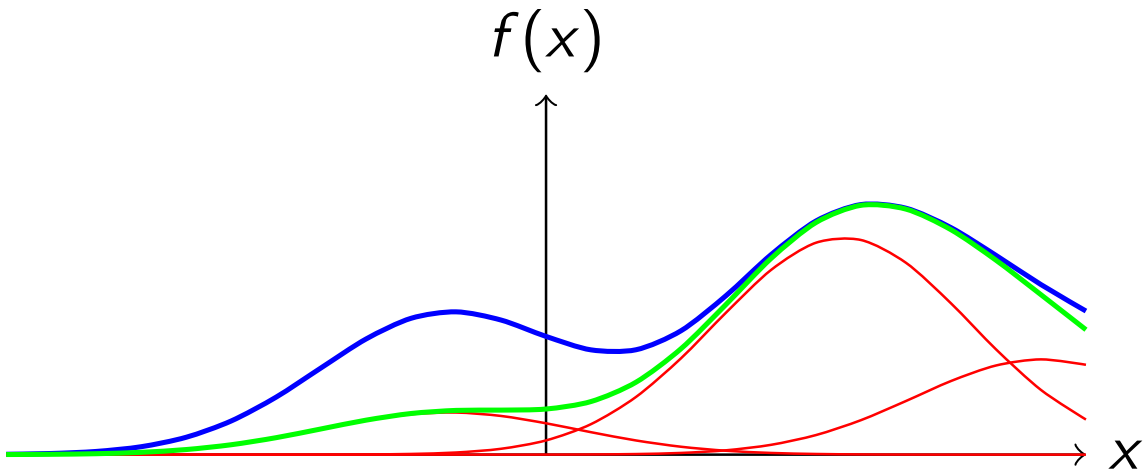

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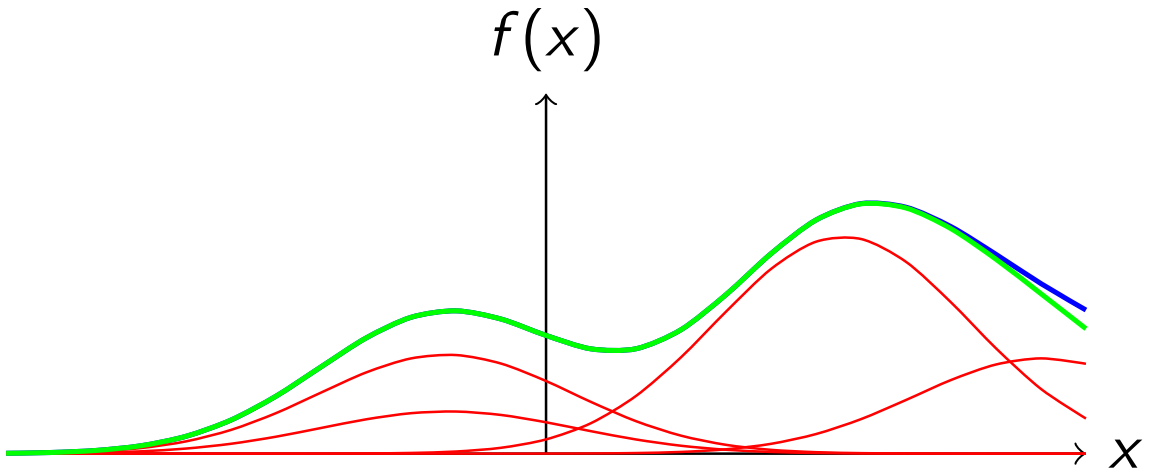

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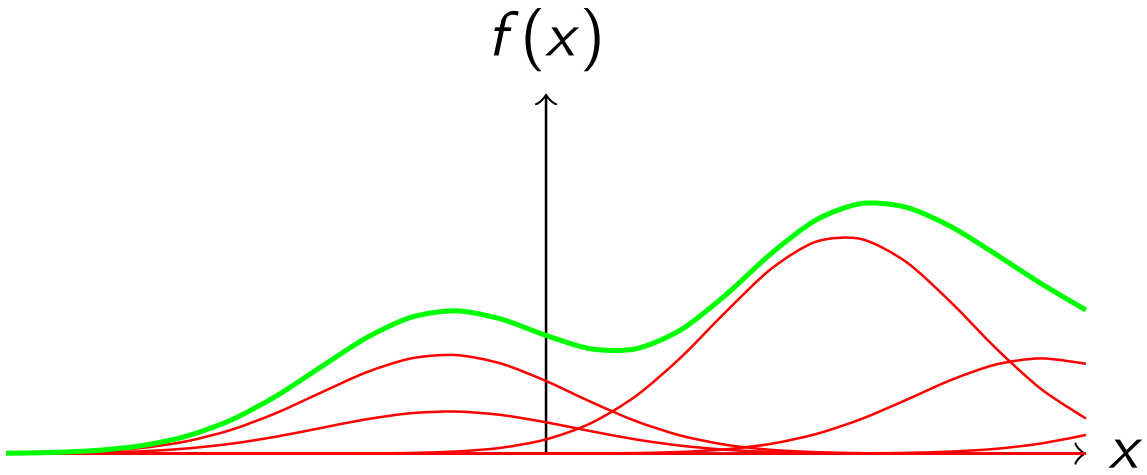

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Learning with Representer Theorem

- Stochastic augmented Lagrangian function of (1) at time t

$$\hat{\mathcal{L}}_t(f,\boldsymbol{\mu}) := \sum_{i\in\mathcal{V}}\Bigg[\ell_i(f_i(\mathbf{x}_{i,t}),y_{i,t}) + \frac{\lambda}{2}\|f_i\|^2_{\mathcal{H}} + \sum_{j\in n_i}\Bigg\{\Big[\mu_{ij}(h_{ij}(f_i(\mathbf{x}_{i,t}),f_j(\mathbf{x}_{i,t})) - \gamma_{ij})\Big] - \frac{\delta\eta}{2}\mu_{ij}^2\Bigg\}\Bigg] \tag{2}$$

where $\boldsymbol{\mu}$ is a lagrange multiplier, with μ_{ij} defined for each $(i,j)\in\mathcal{E}$.

Corollary

Consider the sample average approximation of (1)*, and its associated Lagrangian relaxation. The each ith component of the solution to the resulting saddle-point problem can be expressed as*

$$f_i^* = \sum_{t=1}^{T} w_{i,t}\kappa(\mathbf{x}_{i,t},.) \tag{3}$$

where $w_{i,t}$ are real-valued coefficients.

Functional Derivative

- Functional stochastic gradient of local loss in (2):

$$\ell_i'(f_i(\mathbf{x}_{i,t}), y_{i,t}) := \nabla_{f_i} \ell_i(f_i(\mathbf{x}_{i,t}), y_{i,t})(\cdot) = \frac{\partial \ell_i(f_i(\mathbf{x}_{i,t}), y_{i,t})}{\partial f_i(\mathbf{x}_{i,t})} \frac{\partial f_i(\mathbf{x}_{i,t})}{\partial f_i}(\cdot)$$

- Using the reproducing property of the kernel we obtain

$$\frac{\partial f_i(\mathbf{x}_{i,t})}{\partial f_i} = \frac{\partial \langle f_i, \kappa(\mathbf{x}_{i,t}, \cdot) \rangle_{\mathcal{H}}}{\partial f_i} = \kappa(\mathbf{x}_{i,t}, \cdot) \tag{4}$$

- Now the full gradient result can be written as

$$\begin{aligned} \nabla_{f_i} \hat{\mathcal{L}}_t(f_t, \mu_t) = \ell_i'(f_i(\mathbf{x}_{i,t}), y_{i,t}) \kappa(\mathbf{x}_{i,t}, \cdot) + \lambda f_i \\ + \sum_{j \in n_i} \mu_{ij} h_{ij}'(f_i(\mathbf{x}_{i,t}), f_j(\mathbf{x}_{i,t})) \kappa(\mathbf{x}_{i,t}, \cdot) \end{aligned} \tag{5}$$

Algorithm

- **loop in parallel** for agent $i \in \mathcal{V}$
- Observe local training example realization $(\mathbf{x}_{i,t}, y_{i,t})$
- Send $\mathbf{x}_{i,t}$ to the neighboring nodes, $j \in n_i$ and receive $f_{j,t}(\mathbf{x}_{i,t})$
- Receive $\mathbf{x}_{j,t}$ from the neighbouring nodes, $j \in n_i$ and send $f_{i,t}(\mathbf{x}_{j,t})$
- Stochastic primal descent step on Lagrangian:

$$f_{i,t+1} = f_{i,t}(1-\eta\lambda) - \eta\left[\ell_i'(f_{i,t}(\mathbf{x}_{i,t}), y_{i,t}) + \sum_{j\in n_i} \mu_{ij} h_{ij}'(f_{i,t}(\mathbf{x}_{i,t}), f_{j,t}(\mathbf{x}_{i,t}))\right] \kappa(\mathbf{x}_{i,t}, \cdot) \quad (6)$$

- Stochastic dual ascent step on Lagrangian:

$$\mu_{ij,t+1} = \left[\mu_{ij,t}(1-\delta\eta^2) + \eta\left(h_{ij}(f_{i,t}(\mathbf{x}_{i,t}), f_{j,t}(\mathbf{x}_{i,t})) - \gamma_{ij}\right)\right]_+ \quad (7)$$

Parametric update of weights and dictionary

- Using $f_{i,t}(\mathbf{x}) = \sum_{n=1}^{t-1} w_{i,n}\kappa(\mathbf{x}_{i,n}, \mathbf{x}) = \mathbf{w}_{i,t}^T \boldsymbol{\kappa}_{\mathbf{X}_{i,t}}(\mathbf{x})$, V parallel parametric updates on both kernel dictionaries $\mathbf{X}_i$ and $\mathbf{w}_i$ are

$$\mathbf{X}_{i,t+1} = [\mathbf{X}_{i,t}, \quad \mathbf{x}_{i,t}] \,, \tag{8}$$

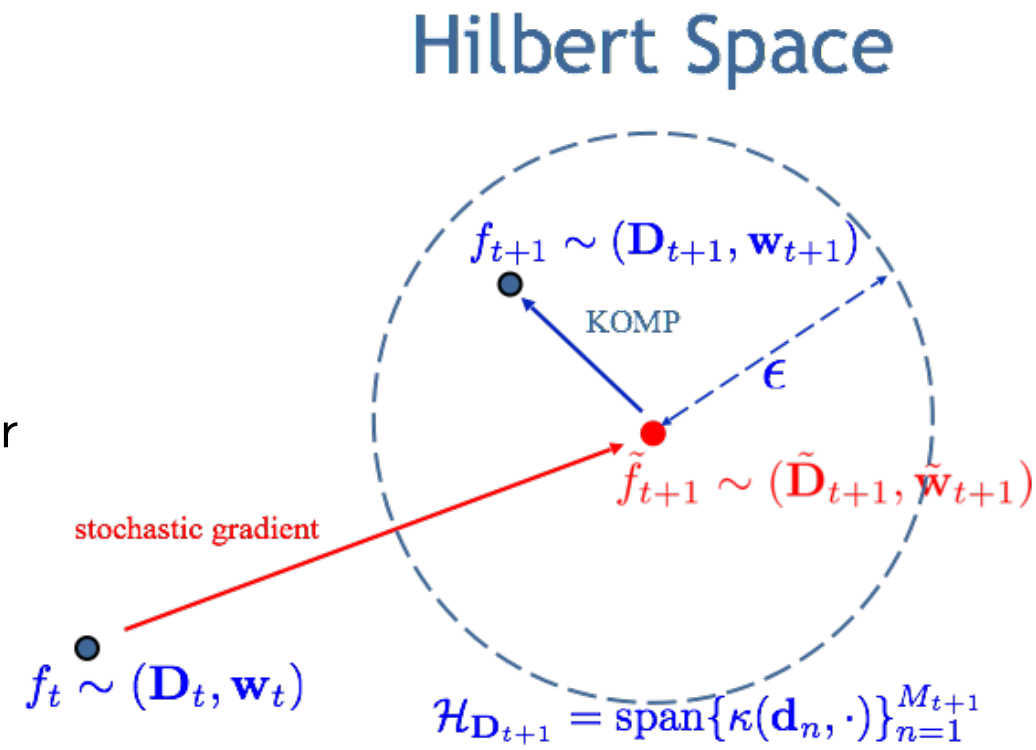
$$[\mathbf{w}_{i,t+1}]_u = \begin{cases} (1-\eta\lambda)[\mathbf{w}_{i,t}]_u, & 0 \le u \le t-1 \\ -\eta\Big(\ell_i'(f_{i,t}(\mathbf{x}_{i,t}), y_{i,t}) \\ \quad + \sum_{j\in n_i} \mu_{ij} h_{ij}'(f_{i,t}(\mathbf{x}_{i,t}), f_{j,t}(\mathbf{x}_{i,t}))\Big), & u = t \end{cases}$$

- Data points $M_{i,t}$ grows by one each time (curse of kernelization).
- **Proj. Funct. Update**: Onto $\mathcal{H}_{\mathbf{D}_{i,t+1}} = \text{span}\{\kappa(\mathbf{d}_{i,n}, \cdot)\}_{n=1}^{M_{t+1}} \subset \mathcal{H}$

$$f_{i,t+1} := \mathcal{P}_{\mathcal{H}_{\mathbf{D}_{i,t+1}}}\left[f_{i,t} - \eta\nabla_{f_i}\hat{\mathcal{L}}_t(f_t, \mu_t)\right]. \tag{9}$$

Kernel Matching Pursuit

- Fix approximation error ϵ
- $\tilde{f}_{t+1} = f_t - \eta \nabla_{f_i} \hat{\mathcal{L}}_t(f_t, \mu_t)$
- Remove kernel element smallest error
- Project $\tilde{f}_{t+1}$ onto resulting RKHS
- Repeat until error is larger than ε



Convergence Analysis

Theorem

Let $S(f_t) := \sum_{i\in\mathcal{V}} \mathbb{E}\left[\ell_i(f_{i,t}(\mathbf{x}_{i,t}), y_{i,t})\right] + \frac{\lambda}{2}\|f_{i,t}\|^2_{\mathcal{H}}$ *as the objective with* f^* *defined in* (1)*. Then with constant step-size* $\eta = 1/\sqrt{T}$ *and approximation budget* $\epsilon_t = \epsilon = P\eta^2$*, where* $P > 0$ *is termed as the parsimony constant.*

$$\sum_{t=1}^{T} \mathbb{E}[S(f_t) - S(f^*)] \leq \mathcal{O}(\sqrt{T}). \tag{10}$$

Furthermore, the time-aggregation of the expected constraint violation grows sub-linearly with iteration T *as*

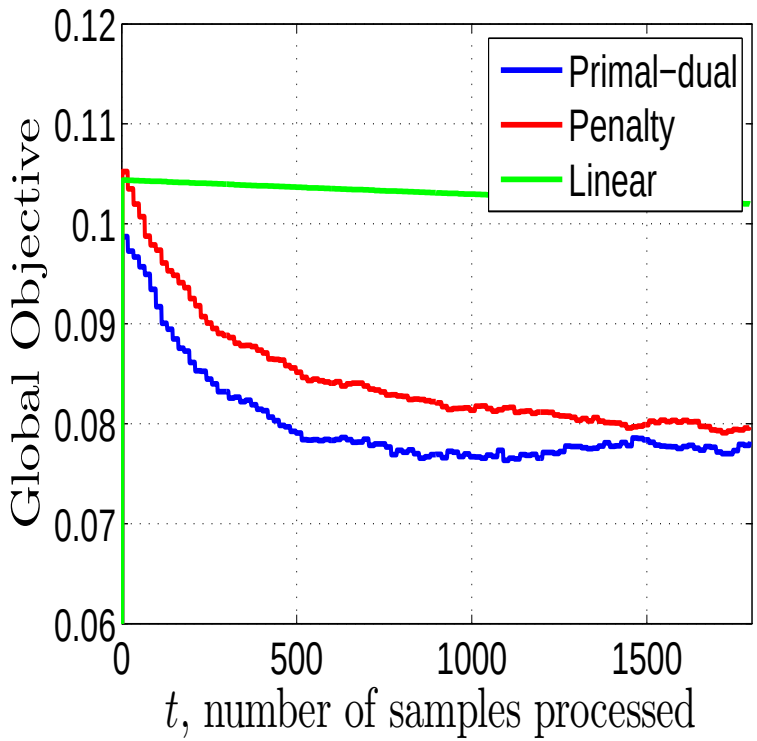
$$\sum_{(i,j)\in\mathcal{E}} \mathbb{E}\left[\sum_{t=1}^{T}(h_{ij}(f_i(\mathbf{x}_{i,t}), f_j(\mathbf{x}_{i,t})) - \gamma_{ij})\right]_+ \leq \mathcal{O}(T^{3/4}). \tag{11}$$

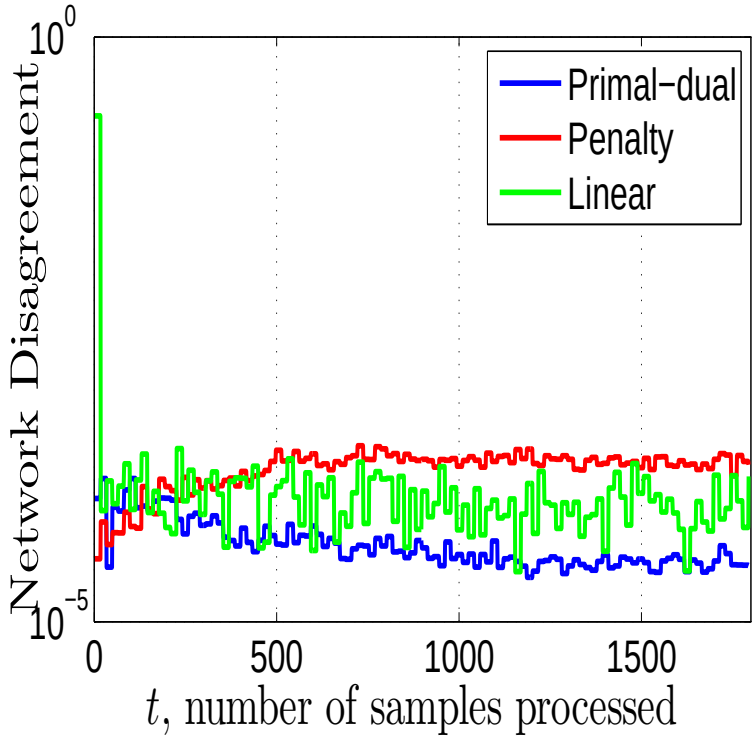
Estimating planar correlated Gaussian random field

- A planar field is a random function of spatial components
- Random field is parameterized by the correlation matrix $\mathbf{R}_x$
  $\Rightarrow [\mathbf{R}_x]_{ij}$ is assumed to have a structure of the form $e^{-\|l_i - l_j\|}$,
- Observation model: $y_{i,t} = h_i \|\mathbf{x}_{i,t}\|^2 + n_{i,t}$, $n_{i,t} \sim \mathcal{N}(0, \sigma^2)$
- 10 nodes distributed in a 10×10 meter square area.
- Instantaneous observation, $\mathbf{x}_t = \pi + \mathbf{C}^T \mathbf{v}_t$
  $\Rightarrow \pi = \{1/V, 2/V, \ldots, 1\}$
  $\Rightarrow \mathbf{C}$: Cholesky factorization of the correlation matrix $\mathbf{R}_x$
  $\Rightarrow \mathbf{v}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{1})$

Field Estimation

- Convergence of global objective and reduced network disagreement





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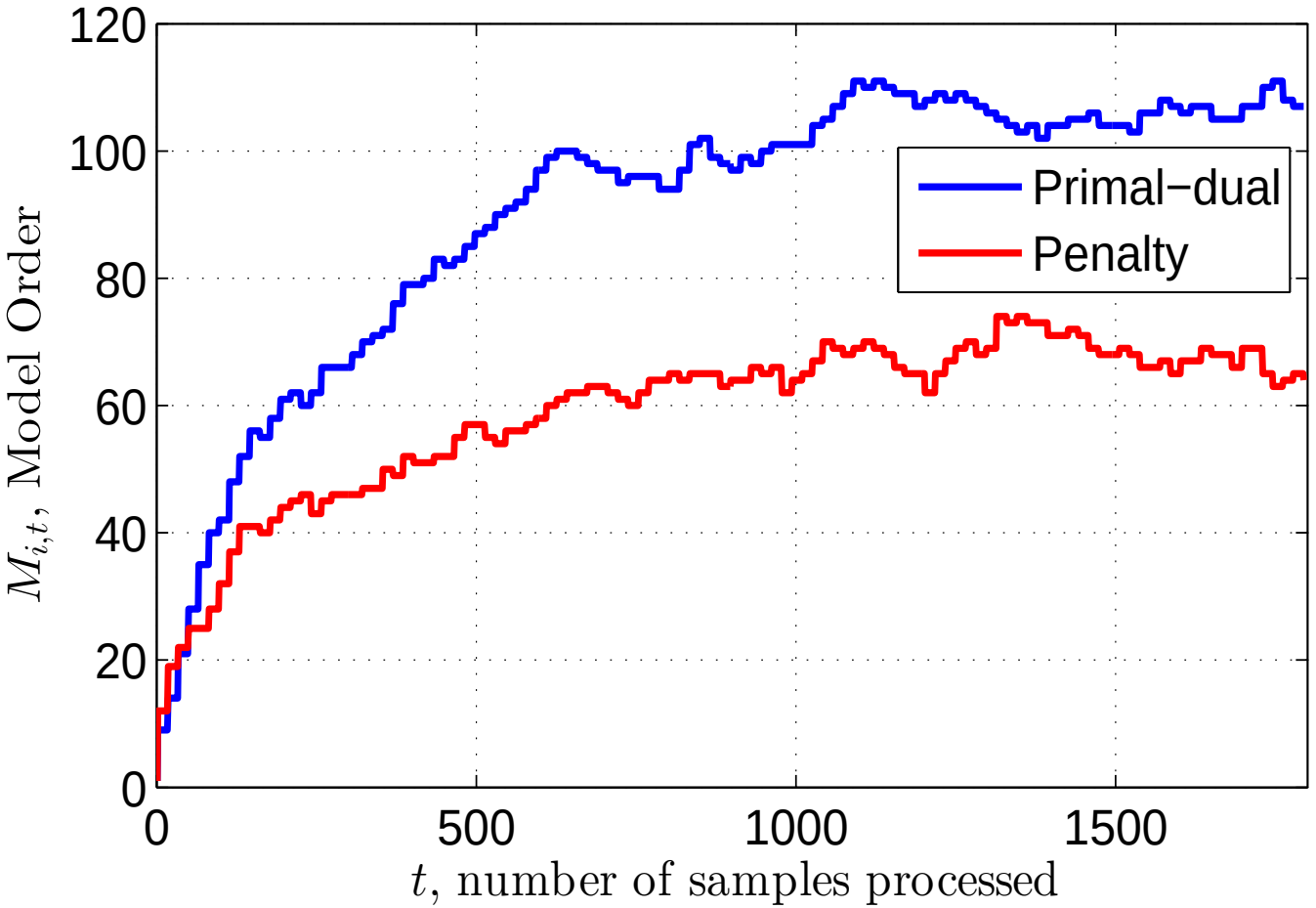

Figure: Model Order $\mathbf{M}_{i,t}$ v.s. Samples

Conclusion and Future Work

- Focused on online learning
  - $\Rightarrow$ Decentralized heterogeneous networks
  - $\Rightarrow$ Non-linear statistical models
- Proposed new variant of projected stochastic primal dual method
  - $\Rightarrow$ Convergence to the optimum
  - $\Rightarrow$ Finite growth of model order
  - $\Rightarrow$ Observed good empirical performance
- **Future Work**:
  - $\Rightarrow$ Asynchrony
  - $\Rightarrow$ Reduce complexity of projections

Appendix: Supervised Learning Background

- $(\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y}$ is random pair $\Rightarrow$ training examples
- $\ell : \mathcal{W} \to \mathbb{R}$ convex loss ($\mathcal{W} \subset \mathbb{R}^p$), merit of statistical model
- Find parameters $\mathbf{w}^* \in \mathbb{R}^p$ that minimize expected risk $L(\mathbf{w})$

$$\mathbf{w}^* := \underset{\mathbf{w}}{\operatorname{argmin}}\, L(\mathbf{w}) := \underset{\mathbf{w}}{\operatorname{argmin}}\, \mathbb{E}_{\mathbf{x},\mathbf{y}}[\ell(\mathbf{w}^\top \mathbf{x}, \mathbf{y})]$$

- Convex Optimization Problem for *linear statistical models*

  $\Rightarrow$ e.g., $y = \mathbf{w}^T \mathbf{x} \in \mathbb{R}$ or $y = \operatorname{sgn}(\mathbf{w}^T \mathbf{x}) \in \{-1, 1\}$
- Solve with favorite descent method $\Rightarrow$ Good Performance

Appendix: Easy to Implement over Networks

- Symmetric, connected and directed network of agents $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- The nodes aims to make inferences from local data
- $|\mathcal{V}| = V$ nodes, $|\mathcal{E}| = M$ edges, and $n_i := \{j : (i,j) \in \mathcal{E}\}$
- Agent $i \in \mathcal{V}$ has a local copy of the classifier $\mathbf{w}_i$

  $\Rightarrow$ Observes some training examples $\Rightarrow (\mathbf{x}_i, \mathbf{y}_i) \in \mathcal{X}_i \times \mathcal{Y}_i$

$$\mathbf{w}^* := \underset{\mathbf{w} \in \mathbb{R}^{p|\mathcal{V}|}}{\operatorname{argmin}} \sum_{i=1}^{|\mathcal{V}|} \mathbb{E}_{\mathbf{x}_i, y_i} \left[\ell(\mathbf{w}_i^\top \mathbf{x}_i, \mathbf{y}_i) \right]$$

$$s.t. \quad \mathbf{w}_i = \mathbf{w}_j \quad \text{for all} \quad j \in n_i$$

- Convex Optimization Problem for *linear statistical models*
- Solve with saddle point algorithms or penalty methods

  $\Rightarrow$ Can be implemented in a distributed fashion

Appendix: How to control the Model order?

- Project (6) onto a lower dimensional subspace $\mathcal{H}_{\mathbf{D}} \subseteq \mathcal{H}$
- $\mathcal{H}_{\mathbf{D}}$ is represented by a dictionary $\mathbf{D} = [\mathbf{d}_1, \ \ldots, \ \mathbf{d}_M] \in \mathbb{R}^{p \times M}$.
- $\mathcal{H}_{\mathbf{D}} = \{f \ : \ f(\cdot) = \sum_{n=1}^{M} w_n \kappa(\mathbf{d}_n, \cdot) = \mathbf{w}^T \boldsymbol{\kappa}_{\mathbf{D}}(\cdot)\}$, $\{\mathbf{d}_n\} \subset \{\mathbf{x}_u\}_{u \leq t}$.
- We denote the un-projected functional update as

$$\tilde{f}_{i,t+1} = f_{i,t} - \eta \nabla_{f_i} \hat{\mathcal{L}}_t(f_t, \mu_t) \ . \tag{12}$$

where $\nabla_{f_i} \hat{\mathcal{L}}_t(f_t, \mu_t) :=$
$\lambda f_{i,t} + \Big[\ell_i'(f_{i,t}(\mathbf{x}_{i,t}), y_{i,t}) + \sum_{j \in n_i} \mu_{ij} h_{ij}'(f_{i,t}(\mathbf{x}_{i,t}), f_{j,t}(\mathbf{x}_{i,t}))\Big] \kappa(\mathbf{x}_{i,t}, \cdot)$.

- $\tilde{f}_{i,t+1}$ in form of dictionary and coefficient vector:

$$\tilde{\mathbf{D}}_{i,t+1} = [\mathbf{D}_{i,t}, \ \ \mathbf{x}_{i,t}] \ ,$$

$$[\tilde{\mathbf{w}}_{i,t+1}]_u = \begin{cases} (1 - \eta\lambda)[\mathbf{w}_{i,t}]_u, & \text{for } 0 \leq u \leq t-1 \\ -\eta\Big(\ell_i'(f_{i,t}(\mathbf{x}_{i,t}), y_{i,t}) + \sum_{j \in n_i} \mu_{ij} h_{ij}'(f_{i,t}(\mathbf{x}_{i,t}), f_{j,t}(\mathbf{x}_{i,t}))\Big), & \text{for } u = t \end{cases}$$